Reciprocal equation:

A reciprocal equation is an equation in which the reciprocal of any root is also a root.

ie for a reciprocal equation $f(x)=0$ if $\alpha_1, \alpha_2, \dots \alpha_n$ are roots then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \dots, \frac{1}{\alpha_n}$ are also the roots of the equation $f(x)=0$

Type: 1

Reciprocal equation of degree 4 with like signs

① Solve the equation $4x^4-20x^3+33x^2-20x+4=0$

Given $4x^4-20x^3+33x^2-20x+4=0$

$\div x^2$

$$\frac{4x^{\cancel{4}2}}{\cancel{x^2}}-\frac{20x^{\cancel{3}}}{\cancel{x^2}}+\frac{33\cancel{x^2}}{\cancel{x^2}}-\frac{20x}{x^2}+\frac{4}{x^2}=0$$

$$4x^2-20x+33-\frac{20}{x}+\frac{4}{x^2}=0$$

$$4\left(x^2+\frac{1}{x^2}\right)-20\left(x+\frac{1}{x}\right)+33=0$$

$$x+\frac{1}{x}=t$$

$$x^2+\frac{1}{x^2}=t^2-2$$

$$4(t^2-2)-20t+33=0$$

$$4t^2-8-20t+33=0$$

$$4t^2-20t+25=0$$

100

$-10 \mid -10$

$$4t^2-10t-10t+25=0$$

$$2t(2t-5)-5(2t-5)=0$$

$$(2t-5)(2t-5)=0$$

$2t-5=0$ $\quad$ $2t-5=0$

$2t=5$ $\quad$ $2t=5$

$\boxed{t=\frac{5}{2}}$ $\quad$ $\boxed{t=\frac{5}{2}}$

$$x+\frac{1}{x}=\frac{5}{2}$$

$$\frac{x^2+1}{x}=\frac{5}{2}$$

$$2(x^2+1)=5x$$

$$2x^2+2=5x$$

$$2x^2-5x+2=0 \qquad \begin{array}{c|c} & 4 \\ -1 & -4 \end{array}$$

$$2x^2-x-4x+2=0$$

$$x(2x-1)-2(2x-1)=0$$

$$(x-2)(2x-1)=0$$

$$x-2=0 \qquad 2x-1=0$$

$$\boxed{x=2} \qquad 2x=1$$

$$\boxed{x=\frac{1}{2}}$$

$\therefore$ The roots are $x=2,\frac{1}{2},2,\frac{1}{2}$

Type : 2

Reciprocal equation of degree 5 with like signs

(2) Solve $x^5-6x^4+7x^3+7x^2-6x+1=0$

Given $x^5-6x^4+7x^3+7x^2-6x+1=0$

$$\begin{array}{c|cccccc} & & & & 7 & -6 & 1 \\ -1 & 1 & -6 & 7 & -14 & 7 & -1 \\ & 0 & -1 & 7 & & & \\ \hline & 1 & -7 & 14 & -7 & 1 & \boxed{0} \end{array}$$

$x=-1$ is a root

$$x^4-7x^3+14x^2-7x+1=0$$

$\div x^2$

$$x^2-7x+14-\frac{7}{x}+\frac{1}{x^2}=0$$

$$\left(x^2+\frac{1}{x^2}\right)-7\left(x+\frac{1}{x}\right)+14=0$$

$$x+\frac{1}{x}=t$$
$$x^2+\frac{1}{x^2}=t^2-2$$
$$(t^2-2)-7t+14=0$$
$$t^2-7t+12=0$$
12
-4 | -3
$$(t-3)(t-4)=0$$
$$t-3=0 \qquad t-4=0$$
$$\boxed{t=3} \qquad \boxed{t=4}$$

case(i)
$$x+\frac{1}{x}=3$$
$$\frac{x^2+1}{x}=3$$
$$x^2+1=3x$$
$$x^2-3x+1=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$a=1,\ b=-3,\ c=1$
$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(1)}}{2(1)}$$
$$x=\frac{3\pm\sqrt{9-4}}{2}$$
$$\boxed{x=\frac{3\pm\sqrt{5}}{2}}$$

case(ii)
$$x+\frac{1}{x}=4$$
$$\frac{x^2+1}{x}=4$$
$$x^2+1=4x$$
$$x^2-4x+1=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$a=1,\ b=-4,\ c=1$
$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}$$
$$x=\frac{4\pm\sqrt{16-4}}{2}$$
$$x=\frac{4\pm\sqrt{12}}{2}$$
$$=\frac{4\pm\sqrt{4\times3}}{2}$$
$$=\frac{4\pm2\sqrt{3}}{2}$$
$$=\cancel{2}\left(\frac{2\pm\sqrt{3}}{\cancel{2}}\right)$$
$$\boxed{x=2\pm\sqrt{3}}$$

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$\therefore$ The roots are $x = 2+\sqrt{3},\ 2-\sqrt{3},\ \frac{3+\sqrt{5}}{2},\ \frac{3-\sqrt{5}}{2},\ -1$

Type: 3

Reciprocal equation of degree 5 with unlike signs.

(3) Solve $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$

Given $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$

$$\begin{array}{c|cccccc} 1 & 1 & -5 & 9 & -9 & 5 & -1 \\ & 0 & 1 & -4 & 5 & -4 & 1 \\ \hline & 1 & -4 & 5 & -4 & 1 & \boxed{0} \end{array}$$

$x = 1$ is a root.

$x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$

$\div x^2$

$x^2 - 4x + 5 - \frac{4}{x} + \frac{1}{x^2} = 0$

$x^2 + \frac{1}{x^2} - 4x - \frac{4}{x} + 5 = 0$

$\left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5 = 0$

$x + \frac{1}{x} = t$

$x^2 + \frac{1}{x^2} = t^2 - 2$

$(t^2 - 2) - 4t + 5 = 0$

$t^2 - 4t + 5 - 2 = 0$

$t^2 - 4t + 3 = 0$ $\quad \begin{array}{c} 3 \\ \hline -1 \mid -3 \end{array}$

$(t-1)(t-3) = 0$

$t - 1 = 0$ | $t - 3 = 0$

$\boxed{t = 1}$ | $\boxed{t = 3}$

Case(i) $t=1$

$$x+\frac{1}{x}=1$$

$$\frac{x^2+1}{x}=1$$

$$x^2+1=x$$

$$x^2-x+1=0$$

$$ax^2+bx+c=0$$

$a=1, b=-1, c=1$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$$

$$=\frac{1\pm\sqrt{1-4}}{2}$$

$$=\frac{1\pm\sqrt{-3}}{2}$$

$$=\frac{1\pm\sqrt{3\times-1}}{2}$$

$$\boxed{x=\frac{1\pm i\sqrt{3}}{2}}$$

Case(ii) $t=3$

$$x+\frac{1}{x}=3$$

$$\frac{x^2+1}{x}=3$$

$$x^2+1=3x$$

$$x^2-3x+1=0$$

$a=1, b=-3, c=1$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(1)}}{2(1)}$$

$$=\frac{3\pm\sqrt{9-4}}{2}$$

$$\boxed{x=\frac{3\pm\sqrt{5}}{2}}$$

$\therefore$ The roots are $x=1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$

Type: 4

Reciprocal equation of degree 6 with unlike signs. (x^3-term absent)

4. Solve $6x^6-35x^5+56x^4-56x^2+35x-6=0$

Given $6x^6-35x^5+56x^4-56x^2+35x-6=0$

1	6	-35	56	0	-56	35	-6
	0	6	-29	27	27	-29	6
-1	6	-29	27	27	-29	6	$\boxed{0}$
	0	-6	35	-62	35	-6	
	6	-35	62	-35	6	$\boxed{0}$	

$x=1,-1$ are the roots

$6x^4-35x^3+62x^2-35x+6=0$

$\div x^2$

$6x^2-35x+62-\frac{35}{x}+\frac{6}{x^2}=0$

$6x^2+\frac{6}{x^2}-35x-\frac{35}{x}+62=0$

$6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62=0$

$x+\frac{1}{x}=t$

$x^2+\frac{1}{x^2}=t^2-2$

$6(t^2-2)-35t+62=0$

$6t^2-12-35t+62=0$

$6t^2-35t+50=0$

$$\begin{array}{c|c} & 300 \\ -15 & -20 \end{array}$$

$6t^2-15t-20t+50=0$

$3t(2t-5)-10(2t-5)=0$

$(2t-5)(3t-10)=0$

$$2t-5=0 \qquad 3t-10=0$$

$$2t=5 \qquad 3t=10$$

$$\boxed{t=\frac{5}{2}} \qquad \boxed{t=\frac{10}{3}}$$

Case(i) $t=\frac{5}{2}$

$$x+\frac{1}{x}=\frac{5}{2}$$

$$\frac{x^2+1}{x}=\frac{5}{2}$$

$$2(x^2+1)=5x$$

$$2x^2+2=5x$$

$$2x^2-5x+2=0 \qquad \frac{4}{-1 \mid -4}$$

$$2x^2-x-4x+2=0$$

$$x(2x-1)-2(2x-1)=0$$

$$(x-2)(2x-1)=0$$

$$x-2=0 \qquad 2x-1=0$$

$$\boxed{x=+2} \qquad 2x=+1$$

$$\boxed{x=+\frac{1}{2}}$$

Case(ii) $t=\frac{10}{3}$

$$x+\frac{1}{x}=\frac{10}{3}$$

$$\frac{x^2+1}{x}=\frac{10}{3}$$

$$3(x^2+1)=10x$$

$$3x^2+3=10x$$

$$3x^2-10x+3=0 \qquad \frac{9}{-1 \mid -9}$$

$$3x^2-x-9x+3=0$$

$$x(3x-1)-3(3x-1)=0$$

$$(x-3)(3x-1)=0$$

$$x-3=0 \qquad 3x-1=0$$

$$\boxed{x=3} \qquad 3x=1$$

$$\boxed{x=\frac{1}{3}}$$

$\therefore$ The roots are $x=1,-1,2,\frac{1}{2},3,\frac{1}{3}$